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Steiner hull algorithm for the uniform orientation metrics

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Abstract

Given a set Z of $n < \infty$ points in the plane and an integer $\lambda \geqslant 2$, we consider the problem of finding a λ-Steiner hull of Z, i.e., a region containing every Steiner minimal tree for Z in the λ-metric. We define a λ-Steiner hull λ-SH(Z) of Z as a set obtained by a maximal sequence of removals of certain open wedge-shaped regions from an initial hull followed by a simplification of its boundary. A perhaps surprising result is presented, namely that a Euclidean MST for Z can be used to decompose the problem of finding λ-SH(Z) into subproblems. Each of these can then be solved recursively using linear searches combined with a sweep line approach. Using this result, we present an algorithm computing λ-SH(Z). This algorithm has $O(\lambda n \log n)$ running time and $O(\lambda n)$ space requirement which is optimal for constant λ. We prove that λ-SH(Z) is independent of the order of removals of the open wedge-shaped regions.

Keywords: Computational geometry; Uniform orientation metric; Steiner tree problem; Steiner hull; Minimum spanning tree

1. Introduction

The classical *Steiner tree problem* is the problem of computing a *Steiner minimal tree* (SMT), i.e., a tree of minimum Euclidean length, spanning a given set of points in the plane [3]. It is distinguished from the minimum spanning tree problem in that new points may be added to shorten the tree. This makes the problem much harder—in fact, it has been shown to be NP-hard.

Steiner minimal trees are useful for routing in VLSI design [4]. Here, an important objective is to interconnect a set of pins on a chip using minimum total wire length. Due to manufacturing limitations however, the orientation of wires have typically been restricted to horizontal and vertical only, making the L_1-metric more suitable for measuring the cost of a network.

More recently, routing using an arbitrary number of uniformly distributed wire orientations has become feasible. For this reason, the *uniform orientation metric* has received some attention in recent years.

This metric is defined as follows. Given an integer $\lambda \geqslant 2$, the set of *uniform orientations* or *λ-orientations* is the set of angles $i\omega$, $i = 0, \ldots, \lambda - 1$, where $\omega = \pi/\lambda$. A line segment, half-line, or line l is said to be *uniformly oriented* if the angle between l and the x-axis is a uniform orientation. The *λ-distance* d_λ between two points is the length of a

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shortest path of uniformly oriented line segments between the points and we refer to d_λ as the λ-*metric* or the *uniform orientation metric*. Note that the 2-metric is the L_1-metric.

A λ-tree is a tree in the plane such that all edges consist of uniformly oriented line segments. The *Steiner tree problem in the uniform orientation metric* (USTP) is to find a λ-tree of minimal length spanning a finite set Z of points or *terminals* in the plane. We refer to such a tree as a *Steiner minimal λ-tree* (λ-SMT) for Z. Additional *Steiner points* may be incorporated to shorten the tree. Like the Euclidean Steiner tree problem, the USTP is NP-hard [2].

In the Euclidean metric, a *Steiner hull* of a given set Z of terminals is a subset of the plane containing every SMT for Z. The convex hull $\mathrm{CH}(Z)$ of Z is an example of a Steiner hull of Z. Having a tight Steiner hull can make the computation of an SMT easier since the number of feasible topologies is reduced as the number of terminals on the boundary of a Steiner hull increases. Furthermore, a non-simple Steiner hull results in the decomposition of an SMT into SMTs for smaller terminal subsets.

Winter [7] presented an $O(n \log n)$ time algorithm for computing a Steiner hull of n terminals. The algorithm starts with $\mathrm{CH}(Z)$ and then iteratively removes certain open wedge-shaped regions to obtain smaller and smaller Steiner hulls.

In this paper, we consider Steiner hulls for the λ-metric. We define a λ-*Steiner hull* of Z to be a subset of the plane known to contain every λ-SMT for Z.

We will address the problem of efficiently finding a tight λ-Steiner hull of Z. We consider a type of λ-Steiner hull, referred to as λ-$\mathrm{SH}(Z)$, which in many ways is similar to that presented in [7] for the Euclidean metric.

We will show that this λ-Steiner hull can be constructed in $O(\lambda n \log n)$ time using $O(\lambda n)$ space and prove that this is optimal under the assumption that λ is a constant. This assumption seems reasonable since in VLSI design, λ is typically much smaller than n (to the author's knowledge, λ-values of 2 and 4 are probably the most widely used today).

The paper is organized as follows. In Section 2, we make various definitions and some simple observations. In Section 3, we prove that a certain set λ-$\mathrm{SH}'(Z)$, from which λ-$\mathrm{SH}(Z)$ is easily derived, is a λ-Steiner hull of Z. Letting n equal the number of terminals, we then present a naive $O((\lambda n)^3)$ time algorithm computing this set. In Section 4, we show how a Euclidean MST can be used to decompose the problem of finding λ-$\mathrm{SH}'(Z)$ into smaller problems each of which can be solved recursively. The results of Section 5 enable us to efficiently check if a region of our partially constructed λ-Steiner hull can be removed. To do this we use a sweep line algorithm for preprocessing. This improves running time to $O((\lambda n)^2)$. In Section 6, we show how to construct λ-$\mathrm{SH}(Z)$ by performing linear searches "in parallel" at each level of the recursion. In Section 7, we show that λ-$\mathrm{SH}(Z)$ can be found in time $O(\lambda n \log n)$ using $O(\lambda n)$ space. We show that this is optimal for constant λ. In Section 8, we prove that λ-$\mathrm{SH}(Z)$ does not depend on the order of removals of open wedge-shaped regions. Finally, we make some concluding remarks in Section 9.

2. Definitions and basic properties

Since we will be dealing with different types of points, we will reserve the letter z for terminals, s for Steiner points, u, v, and w for vertices (terminals and Steiner points), and other letters for regular points.

Let p and q be two points in the plane. If pq is uniformly oriented, there is a unique shortest path from p to q in the λ-metric, namely the line segment pq. Otherwise, the set of shortest paths from p to q in the λ-metric constitutes a parallelogram $prqr'$. The shortest paths prq and $pr'q$ from p to q are called the *critical paths* from p to q and r and r' are called *corner points* of the critical paths.

The λ-*lune of p and q* denoted $L_\lambda(p,q)$ is defined as the set $L_\lambda(p,q) = \{s \in \mathbb{R}^2 \mid d_\lambda(s,p) < d_\lambda(p,q) \wedge d_\lambda(s,q) < d_\lambda(p,q)\}$, see Fig. 1.

If a, b, and c are three distinct points in the plane then we define $\angle abc$ as the smaller non-negative angle between line segments ba and bc.

Let l be a half-line emanating from a point p and let l_x be the horizontal half-line emanating from p and lying to the right of p. Then we say that l has *direction* $\theta \in [0, 2\pi[$ if the counter-clockwise angle from l_x to l equals θ.

Given a simple polygon P, we define a *clockwise walk of P* to be a walk of the boundary of P such that the interior of P is to the right during the walk. For a tree T embedded in the plane, consider inflating its edges. An outer walk of T is then called *clockwise* if the "interior" of T is to the right during the walk.

For any subset X of $\mathbb{R}^2$ we let X° denote the interior of X. We shall assume that all subsets of the plane considered in this paper are closed unless otherwise stated.

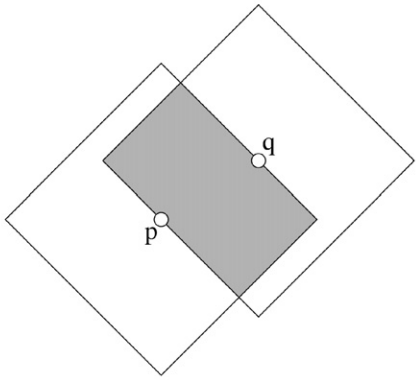

Fig. 1. The λ-lune $L_\lambda(p,q)$ (dark area) of points p and q, here shown for $\lambda = 2$.

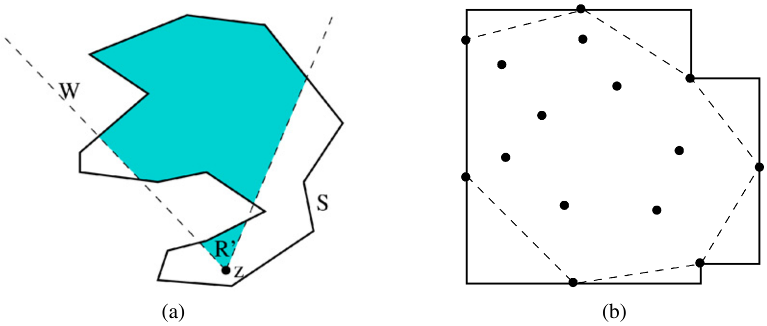

Fig. 2. (a) $W \cap S$ is the union of simple polygons. (b) λ-CH(Z), here shown for $\lambda = 2$. The dashed polygon shows CH(Z).

Suppose that l_a and l_b are uniformly oriented half-lines emanating from a common point p. Then we let $W(l_a, l_b)$ denote the open wedge-shaped region of points hit when sweeping a halfline emanating from p counter-clockwise from l_a to l_b. Halfline l_a is called the *right leg* and l_b is called the *left leg* of $W(l_a, l_b)$. Let θ denote the counter-clockwise angle from l_a to l_b. If $\theta = \lfloor 2\lambda/3 \rfloor \omega$ then $W(l_a, l_b)$ is called a λ-*wedge* (*of* p) and if $\theta = \omega$, $W(l_a, l_b)$ is called a λ-*cone* (*of* p). If $p_a \neq p$ is a point on l_a and $p_b \neq p$ is a point on l_b then we define $W(p, p_a, p_b) = W(l_a, l_b)$.

The λ-Steiner hulls that we will consider in this paper are constructed by iteratively removing regions bounded by λ-wedges from an initial hull. We need to make sure that each such region does not contain any part of any λ-SMT for Z. In particular it should not contain any terminals.

This motivates the following definition. Let S be a simple polygon and let W be a λ-wedge of a terminal $z \in S$. Then $W \cap S$ is a union of regions bounded by simple polygons. One of these regions, say R', contains z on its boundary, see Fig. 2(a). Suppose that $R = W^\circ \cap R'$ is non-empty and contains no terminals of Z. Then W is called *safe* (*in* S), R is called a *safe region* (*of* S) and the removal of R from S is called a *safe removal* (*from* S). We say that R is *bounded by* W.

We refer to a subpath of the boundary of S connecting two consecutive terminals as a *boundary subpath* (*of* S). If R is a safe region of S then the part I of the boundary of S intersecting R is free of terminals and thus I is fully contained in a boundary subpath p. Let z_1 be the terminal of the safe λ-wedge bounding R and let z_2 and z_3 be the end terminals of p. Then we say that R is *bounded* by z_1, z_2, and z_3 and we refer to z_1 as the *base terminal* of R.

Let $z_0, \ldots, z_{r-1}$ be a cyclic ordering of the terminals on the boundary of the convex hull CH(Z) of Z. For $i = 0, \ldots, r-1$, let P_i be the parallelogram consisting of all the shortest paths between z_i and z_{i+1} in the λ-metric (indices are modulo r). The *λ-convex hull* λ-CH(Z) of Z is then defined as $\lambda\text{-CH}(Z) = \text{CH}(Z) \cup \bigcup_{i=0}^{r-1} P_i$, see Fig. 2(b).

In the following, we will let λ-SH$'(Z)$ denote a set obtained by a maximal sequence of safe removals from the initial hull λ-CH(Z).

Note that for each set S obtained in such a maximal sequence, all concave angles of the boundary of S are at terminals. This implies that all safe regions removed are convex. Also note that the line segments bounding these regions are all uniformly oriented.

We obtain λ-SH(Z) from λ-SH$'(Z)$ by replacing each boundary subpath of λ-SH$'(Z)$ by a critical path between the two terminals defining the endpoints of that boundary subpath; the critical paths are chosen such that all corner points are right turns in a clockwise walk of λ-SH(Z). If the line segment l between the two terminals is uniformly oriented, the boundary subpath is replaced by l.

As we shall see, λ-SH(Z) is a λ-Steiner hull and it is independent of the chosen maximal sequence of safe removals.

3. λ-Steiner hull

For now, let us consider λ-SH$'(Z)$. We will return to λ-SH(Z) in Section 6.

In this section, we prove that λ-SH$'(Z)$ is a λ-Steiner hull of Z. We do this by showing that λ-CH(Z) is a λ-Steiner hull of Z and that each safe removal does not cut off any part of any λ-SMT T for Z. The former is shown in Lemma 4 below. To show the latter we will show that after a safe removal,

(1) no terminal is cut off;
(2) no Steiner point of T is cut off;
(3) no part of any edge of T is cut off.

The first part follows by definition of a safe λ-wedge. The second part is shown in Lemma 2 below and the third part in Lemma 3. We need the following result.

Lemma 1. *Let (u, v) be any edge of a λ-SMT. No vertex of the λ-SMT can lie in the λ-lune $L_\lambda(u, v)$.*

Proof. If w is a vertex in $L_\lambda(u, v)$, we may assume that the λ-SMT contains a path from w to u not containing v. Since $d_\lambda(w, v) < d_\lambda(u, v)$, the λ-SMT can be shortened by deleting (u, v) and adding (w, v), a contradiction. □

Lemma 2. *Let S be a λ-Steiner hull of terminal set Z and let S' be the set obtained by a safe removal from S. Then all Steiner points of any λ-SMT for Z belong to S'.*

Proof. Let W be a safe λ-wedge of a terminal z and let T be a λ-SMT for Z. Suppose for the sake of contradiction that the safe region R of S bounded by W contains a Steiner point s of T. Pick s such that its Euclidean distance to z is maximized over all Steiner points of T contained in R. Since the angle between the legs of W is $\lfloor 2\lambda/3 \rfloor\omega$ and since the angle between Steiner tree edges of s is at most $(\lfloor 2\lambda/3 \rfloor + 1)\omega$ [1] there exists an edge (s, v) in T such that $v \in W$. Since S is a λ-Steiner hull of Z, (s, v) is fully contained in S and since $s \in R$, we have $v \in R$.

By the choice of s, v must be a terminal. But this contradicts the assumption that R is a safe region. □

Lemma 3. *Let S be a λ-Steiner hull of terminal set Z and let S' be the set obtained by a safe removal from S. Then all edges of any λ-SMT for Z are fully contained in S'.*

Proof. Let $W(l_1, l_2)$ be a safe λ-wedge of a terminal z and let T be a λ-SMT for Z. We claim that the safe region R bounded by $W(l_1, l_2)$ does not intersect any edge of T.

Assume, for the sake of contradiction, that (u, v) is an edge of T intersecting R. By Lemma 2, (u, v) must cross $W(l_1, l_2)^\circ$. Without loss of generality, assume that u belongs to the halfplane of the line through l_1 not containing l_2, see Fig. 3.

Suppose that the line segment from z to u makes angle θ_u with the x-axis, that the line segment from z to v makes angle θ_v with the x-axis, that l_1 makes angle θ_1 with the x-axis, and that l_2 makes angle θ_2 with the x-axis. By rotating about z by a multiple of ω if necessary, we may assume that $0 \leqslant \theta_u < \omega$ and we have the inequalities $\theta_u \leqslant \theta_1 < \theta_2 \leqslant \theta_v$.

Let C_u be the set of points having λ-distance at most $d_\lambda(u, z)$ to u. We assume that $\theta_u > 0$. The case $\theta_u = 0$ is handled similarly. Since (u, v) crosses $W(l_1, l_2)^\circ$ we have $\theta_v < \pi + \omega$. The intersection of C_u and the λ-cone of u containing z is a triangle $\triangle uab$ and line segment ab makes angle $\lceil \frac{\theta_u}{\omega} \rceil\omega + \frac{\pi-\omega}{2}$ with the x axis. Since

$$\left\lceil \frac{\theta_u}{\omega} \right\rceil \omega + \frac{\pi - \omega}{2} < \theta_2 \leqslant \theta_v < \pi + \omega = \left\lceil \frac{\theta_u}{\omega} \right\rceil \omega + \pi < \left\lceil \frac{\theta_u}{\omega} \right\rceil \omega + \frac{\pi - \omega}{2} + \pi,$$

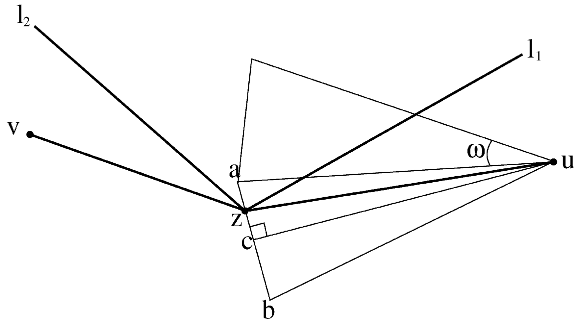

Fig. 3. The situation in the proof of Lemma 3.

and since $z \in ab$, v must belong to the halfplane of the line through ab not containing u. Since C_u is convex, $v \notin C_u$ implying that $d_\lambda(u, v) > d_\lambda(u, z)$. Symmetrically, $d_\lambda(u, v) > d_\lambda(v, z)$. Hence, z belongs to $L_\lambda(u, v)$ contradicting Lemma 1. □

By applying Lemmas 2 and 3 to a λ-Steiner hull containing λ-CH(Z) (pick say the entire plane) it is easy to show the following.

Lemma 4. *λ-CH(Z) is a λ-Steiner hull of Z.*

We have now shown the main result of this section.

Theorem 5. *λ-SH$'(Z)$ is a λ-Steiner hull of Z.*

A naive way of computing λ-SH$'(Z)$ is as follows. First we initialize $S =$ λ-CH(Z). Then for each terminal in S and each λ-wedge W of z, we check if W bounds a safe region by computing the simple polygon R' of $W \cap S$ containing z and checking each terminal for inclusion in $R = R' \cap W^\circ$. If R contains no terminals, we set $S := S \setminus R$ and repeat the algorithm on S.

Recalling that a safe region is convex with a boundary consisting of uniformly oriented line segments, it can be determined whether a region is safe in $O(\lambda n)$ time. Since a terminal can be a base terminal $O(\lambda)$ times throughout the course of the algorithm and since there are $O(\lambda n)$ candidate safe regions in each iteration, it follows that the above algorithm can be implemented to run in $O((\lambda n)^3)$ time using $O(n)$ space. We will show how to find λ-SH$'(Z)$ more efficiently.

4. MST regions

Let Z be a terminal set. In the following, let M denote a fixed Euclidean MST for Z. The boundary subpaths of λ-CH(Z) together with the edges of M partition λ-CH(Z) into faces or *MST regions*.

We will show that computing λ-SH$'(Z)$ can be restricted to each MST region. The following lemma will prove useful.

Lemma 6. *$M \subseteq$ λ-SH$'(Z)$.*

Proof. We show that if S is any partially constructed λ-SH$'(Z)$ then S contains M. The proof is by induction on the number $r \geqslant 0$ of safe regions removed. Since $M \subseteq$ CH$(Z) \subseteq$ λ-CH(Z), this holds when $r = 0$.

Now suppose that after the removal of r safe regions, $M \subseteq S$. For the sake of contradiction, suppose there is a safe region R such that $M \nsubseteq S \setminus R$. Let z be the base terminal of R and suppose that, looking from z, r_1 respectively r_2 is the first point of intersection between the boundary of S and the left respectively right half-line of the safe λ-wedge bounding R.

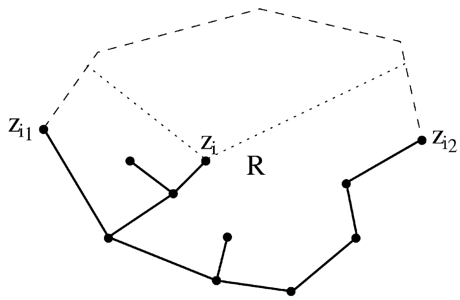

Fig. 4. Removing a safe region splits a subregion into two smaller subregions.

Since R contains no terminals, there must be an edge e of M crossing R. By the induction hypothesis, e must cross zr_1 in a point p and zr_2 in a point q. Let z_1, z_2 be end terminals of e such that $z_1 p$ and $q z_2$ are not contained in R.

If we remove e from M we split M into two components one containing say z_1 and z and the other containing z_2. Since $\angle pzq \geqslant \frac{\pi}{2}$, we also have $\angle z_1 z z_2 \geqslant \frac{\pi}{2}$, implying that $|e| = |z_1 z_2| > |z_2 z|$. Thus, by adding edge (z_2, z), a new tree M' spanning Z is obtained and $|M'| < |M|$, a contradiction. Thus $M \subseteq S \setminus R$. $\square$

Consider a clockwise walk of M visiting the terminals of an MST region R_{MST} in the order $z_0, \ldots, z_m$. A terminal may appear several times in this list since it may be visited more than once. Now consider a safe region R of λ-CH(Z) bounded by z_0, z_m and a base terminal z. By Lemma 6, R is fully contained in R_{MST}, hence $z = z_i$ for some $i \in \{0, \ldots, m\}$. The removal of R separates R_{MST} into a subregion containing the terminals $z_0, \ldots, z_i$ and a subregion containing the terminals $z_i, \ldots, z_m$. Generalizing, we have

Theorem 7. *Consider a subregion induced by terminals $z_{i_1}, \ldots, z_{i_2}$. If a safe region is bounded by z_{i_1}, z_{i_2}, and some base terminal z_i then $z_i \in \{z_{i_1}, \ldots, z_{i_2}\}$. The removal of this safe region partitions the subregion into two smaller subregions, one containing $z_{i_1}, \ldots, z_i$ and one containing $z_i, \ldots, z_{i_2}$* (*Fig.* 4).

Theorem 7 yields a recursive algorithm that removes safe regions from an MST region. Unfortunately, since we do not yet have a strategy for searching for base terminals, this result alone does not improve the $O((\lambda n)^3)$ asymptotic running time of our brute-force algorithm from Section 3. However, in Section 6 we shall present a clever strategy for finding base terminals.

5. Finding safe regions

Let M and R_{MST} be defined as in the preceding section. In this section we will show that, given a λ-wedge of a terminal of (a subregion of) R_{MST}, we can determine whether this λ-wedge bounds a safe region in constant time with $O(\lambda n \log n)$ preprocessing time. The idea is to use the fact that terminals in R_{MST} are all on the same path in the MST. Thus, instead of checking each terminal for inclusion in a candidate safe region, we simply check if the path crosses the boundary of that region. To do this efficiently, we will need the following definitions.

Let $z_k \notin \{z_0, z_m\}$ be a terminal of R_{MST}. Let $d \in \{0, \ldots, 2\lambda - 1\}$ and let l be the half-line emanating from z_k with direction $d\omega$. If $e = (z_i, z_{i+1})$ is an edge of R_{MST} we say that *e is d-visible from z_k* if l avoids edges and terminals of R_{MST} before intersecting e in its interior looking from z_k, see Fig. 5. If $z_j \neq z_k$ is a terminal of R_{MST} we say that *z_j is d-visible from z_k* if l avoids edges and terminals of R_{MST} before intersecting z_j looking from z_k.

Let R be a subregion of R_{MST} induced by terminals $z_{i_1}, \ldots, z_{i_2}$ and suppose that $z_k \in R$. To simplify the analysis, we assume that $z_k \notin \{z_{i_1}, z_{i_2}\}$; the case $z_k \in \{z_{i_1}, z_{i_2}\}$ is handled in a similar way. We make the following simple observations.

Any edge of R (i.e., an edge (z_i, z_j) with $i_1 \leqslant i, j \leqslant i_2$) has an oppositely directed edge in R if and only if it does not belong to the boundary of R, see Fig. 6. We say that an edge respectively terminal on the boundary of R *bounds* R. If z_k has an ingoing edge e in R that bounds R, this edge is unique and we refer to the endpoint of e opposite z_k as $in(z_k)$. We define $out(z_k)$ similarly.

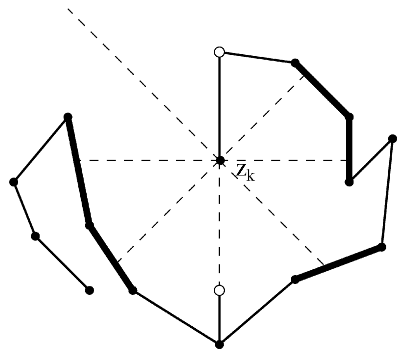

Fig. 5. Edges and terminals d-visible from z_k for $d = 0, \dots, 7$ and $\lambda = 4$. Bold edges and white terminals are d-visible from z_k.

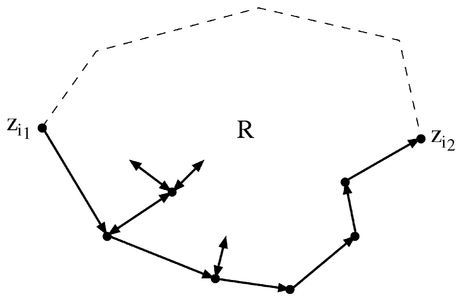

Fig. 6. An edge bounds R if and only if it has no oppositely directed edge. The boundary subpath between z_{i_1} and z_{i_2} is shown as dashed line segments.

As above, let l be the half-line emanating from z_k with direction $d\omega$. Let $e = (z_i, z_{i+1})$ be an edge of R which is d-visible from z_k and let l' be the line through e. Imagine walking along l starting at z_k. Then we cross e from the outside of R if and only if e bounds R and z_k belongs to the right halfplane of l' looking from z_i to z_{i+1}. If a terminal $z_j \notin \{z_{i_1}, z_{i_2}\}$ of R is d-visible from z_k then we cross z_j from the outside of R if and only if $in(z_j)$ and $out(z_j)$ exist and z_k belongs to $W(z_j, in(z_j), out(z_j))^\circ$.

Theorem 9 below relates the above definitions to safe regions. We need the following lemma.

Lemma 8. *With the above definitions, suppose that z_k is a base terminal of a safe region of R. If an edge e of R is d-visible from z_k then halfline l crosses e from the outside of R looking from z_k. If a terminal $z_j \notin \{z_{i_1}, z_{i_2}\}$ in R is d-visible from z_k then l crosses z_j from the outside of R looking from z_k.*

Proof. We only show the first part of the lemma. The second part is shown similarly. Let e be an edge of R which is d-visible from z_k. Suppose for the sake of contradiction that l does not cross e from the outside of R looking from z_k. Since l bounds a safe region with base terminal z_k, l must intersect the boundary subpath P between z_{i_1} and z_{i_2} (at least) twice since we leave R and then enter R again when moving from z_k to e. Let p respectively q be the first respectively second such intersection when looking from z_k. Let $p_0 = z_{i_1}$, $p_{r+1} = z_{i_2}$ and let $p_1, \dots, p_r$ be the corner points of P when moving from z_{i_1} to z_{i_2}. Pick r_1 and r_2 such that p belongs to $p_{r_1}p_{r_1+1}$ and such that q belongs to $p_{r_2}p_{r_2+1}$.

Since the only concave angles of the boundary of the current λ-Steiner hull are at terminals, we must have the situation depicted in Fig. 7. Since no edges or terminals of R_{MST} belong to $P \setminus \{z_{i_1}, z_{i_2}\}$, it follows that R_{MST} is contained in polygon $S = pp_{r_1+1}p_{r_1+2} \cdots p_{r_2}qp$ and thus isolated from the rest of the MST M, a contradiction. □

We are now ready for the main result of this section. Theorem 9 below gives necessary and sufficient conditions for two half-lines to bound a safe region. The theorem assumes that terminals are in *general position*, which in this setting means that no two terminals are on the same uniformly oriented line. See Appendix A for details about how this restriction can be removed.

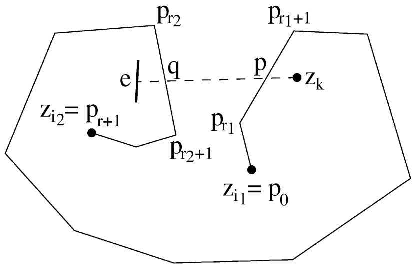

Fig. 7. The situation leading to a contradiction in the proof of Lemma 8.

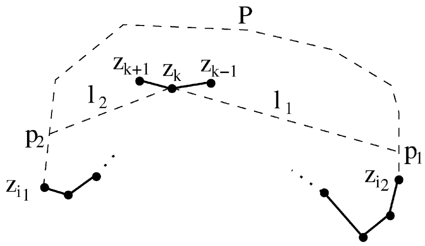

Fig. 8. The impossible situation in the proof of Theorem 9.

Theorem 9. *Let R and $z_k \neq z_{i_1}, z_{i_2}$ be defined as above. Let l_1 and l_2 be half-lines emanating from z_k and having directions $d_1\omega$ and $d_2\omega$ such that $W(l_2, l_1)$ is a λ-wedge. Then the following two statements are equivalent.*

(1) *$W(l_2, l_1) \subseteq W(z_k, z_{k+1}, z_{k-1})$ and for $m = 1, 2$, if an edge e in R is d_m-visible from z_k then l_m crosses e from the outside of R looking from z_k.*
(2) *$W(l_2, l_1)$ is a safe λ-wedge in R.*

Proof. Assume that (1) is satisfied. For $m = 1, 2$, let p_m be the first intersection point between $l_m \setminus \{z_k\}$ and an edge of R, a terminal of R, or the boundary subpath P between z_{i_1} and z_{i_2}. The existence of p_m follows from the assumption $W(l_2, l_1) \subseteq W(z_k, z_{k+1}, z_{k-1})$.

Since terminals are in general position, p_m cannot be a terminal. If p_m belonged to the interior of an edge $e = (z_j, z_{j+1})$ of R then e would be d_m-visible from z_k implying that l_m would cross e from the outside of R looking from z_k. But $z_k p_m \subseteq R$. For $m = 1, 2$ we conclude that p_m belongs to P.

We also need to check that, when moving from z_{i_1} to z_{i_2} along P, we first pass p_1 and then p_2. If we assume the opposite (see Fig. 8) then, since edges (z_{k-1}, z_k) and (z_k, z_{k+1}) are consecutive in the path of edges from z_{i_1} to z_{i_2} in R and terminals are in general position, there can be no path of edges connecting z_k and z_{i_1}, a contradiction.

The above shows that no edges cross the boundary of the candidate safe region R' bounded by $W(l_2, l_1)$. Thus, R' contains no terminals and so $(1) \Rightarrow (2)$. The other implication follows from Lemma 8. □

We find d-visible edges and terminals in R_{MST} using an $O(n \log n)$ time and $O(n)$ space sweep line algorithm for each of the 2λ values of d. The algorithm is straightforward and so we will not discuss it further.

Using Theorem 9 we can now determine in $O(1)$ time whether a terminal is a base terminal in (a subregion of) R_{MST} once d-visible edges and terminals in R_{MST} have been found. This improves the running time of our algorithm to $O((\lambda n)^2)$.

We will now turn our attention to λ-SH(Z) and show how to compute this set using only $O(\lambda n \log n)$ time and $O(\lambda n)$ space.

6. Parallel linear searches

In this section, we will describe an efficient way of searching through the terminals of a (subregion of) R_{MST}.

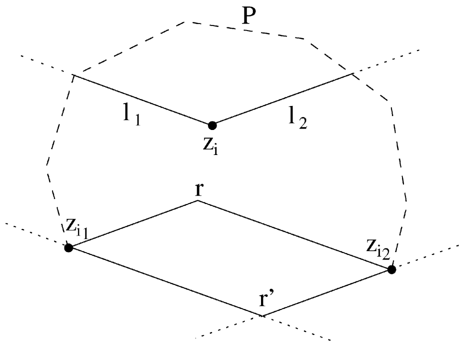

Fig. 9. The regions considered in the proof of Theorem 10.

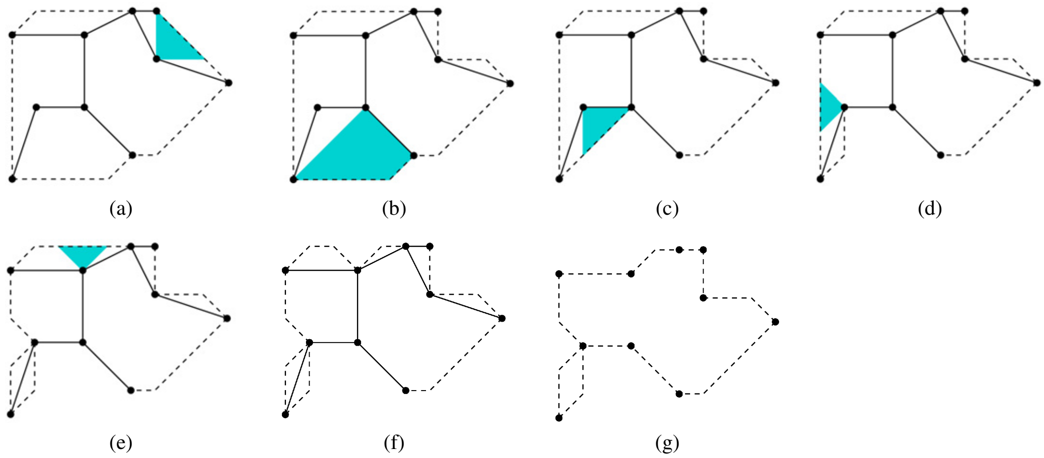

Fig. 10. Illustrating the algorithm for $\lambda = 4$. (a) Initial λ-Steiner hull λ-CH(Z). (b)–(f) Successive removals of safe regions. (g) λ-SH(Z). MST edges respectively boundary subpaths shown as solid respectively dashed line segments. Note that boundary subpaths are not maintained in the actual algorithm.

Recall that, in order to compute λ-SH(Z), we do not need the boundary subpaths of λ-SH$'(Z)$ but only the terminals on the boundary of λ-SH$'(Z)$ and the order in which they occur. Theorem 9 shows that we do not need to maintain the boundary subpaths throughout the course of the algorithm.

With the above in mind and using the recursive algorithm of Section 4, the overall algorithm that removes safe regions in R_{MST} is as follows. For any subregion R induced by terminals $z_{i_1}, \ldots, z_{i_2}$ we find a base terminal in R and recursively remove safe regions in the two new subregions. If no base terminal is found we terminate.

We check for base terminals in the order $z_{i_1+1}, z_{i_2-1}, z_{i_1+2}, z_{i_2-2}, \ldots$. In effect, we perform two linear searches in parallel, one visiting terminals in the order $z_{i_1+1}, z_{i_1+2}, \ldots$ and one visiting the terminals in the order $z_{i_2-1}, z_{i_2-2}, \ldots$. The idea is that a long search time is compensated for by an even (good) split of the subregion (or termination if no base terminal exists) whereas an uneven (bad) split is compensated for by a short search time.

Note that we no longer check if z_{i_1} and z_{i_2} are base terminals since safe regions with either of these terminals as base terminals would only affect the boundary subpaths.

Fig. 10 illustrates the various steps of the algorithm on an instance consisting of ten terminals. Before proving time and space bounds for this algorithm, we need the following theorem which shows that λ-SH(Z) is in fact a λ-Steiner hull of Z.

Theorem 10. *The set λ-SH(Z) is a λ-Steiner hull of Z.*

Proof. Let R be a subregion of an MST region in λ-SH$'(Z)$ and let $z_{i_1}, \ldots, z_{i_2}$ be the terminals of R. Let $z_{i_1} r z_{i_2}$ be the critical path from z_{i_1} to z_{i_2} making a right turn at r and let $z_{i_1} r' z_{i_2}$ be the critical path from z_{i_1} to z_{i_2} making a left turn at r'. Let P be the boundary subpath in λ-SH$'(Z)$ between z_{i_1} and z_{i_2}.

We claim that no terminal belongs to the interior of the bounded region R' bounded by P and $z_{i_1}r'z_{i_2}$. For suppose that z_i is such a terminal. Let l_1 respectively l_2 be the half-line emanating from z_i having the same direction as the half-line emanating from r' and intersecting z_{i_1} respectively z_{i_2}, see Fig. 9.

We may assume that z_i is the only terminal in $W(l_2, l_1) \cap R'$. But then $W(l_2, l_1)$ contains a safe region with base terminal z_i, contradicting the fact that no safe regions can be removed from λ-SH(Z). We conclude that the interior of R' contains no terminals.

Now let R'' be the bounded region bounded by P and $z_{i_1}rz_{i_2}$. We will show that the interior of R'' contains no part of any λ-SMT for Z. By the above, the interior of R'' contains no terminals of Z and by using a similar argument as in Lemma 2, it follows that R'' contains no Steiner points of any λ-SMT for Z. It is then easy to see that no edges of any λ-SMT for Z intersect the interior of R''.

Applying the above to each subregion of λ-SH$'(Z)$ shows the theorem. □

7. Running time and space requirement

In this section, we show that our algorithm computing λ-SH(Z) has worst-case running time $O(\lambda n \log n)$ and $O(\lambda n)$ space requirement where n is the number of terminals. We show that, regarding λ as constant, this is optimal.

Theorem 11. *The algorithm presented above has $O(\lambda n \log n)$ worst-case running time.*

Proof. We can find CH(Z), M, and the MST regions of M in $O(n \log n)$ time. Consider any MST region R and let r be the number of terminals (with repetitions) on the subpath in R induced by the clockwise walk of M. We need to show that it takes $O(\lambda r \log r)$ time to remove safe regions from R.

Since we make 2λ calls to the sweep line algorithm, the total time spent on this is $O(\lambda r \log r)$. Now let $t(k)$ denote the highest number of terminals checked in any subregion R' of R containing exactly k terminals. Here we also count terminals checked in recursive calls to subregions of R'. We claim that

$$t(k) \leqslant (2k-3)\lg k - 1. \tag{1}$$

We show (1) by induction on $k \geqslant 2$. The base case is trivial since then we perform no checks. Now let $k > 2$ and assume that (1) holds for all values smaller than k. To show (1) for k, suppose first that R' contains no base terminals. Then we search through $k-2$ terminals before terminating, i.e., we search through $k-2$ terminals.

Now suppose instead that we find a base terminal in R' after having checked i terminals. Since we search in parallel from both ends of the path in R', we split R' into one subregion containing $\lfloor (i+1)/2 \rfloor + 1$ terminals and one subregion containing $k - \lfloor (i+1)/2 \rfloor$ terminals.

By the above,

$$t(k) \leqslant \max\Big\{k-2, \max_{i=1,\dots,k-2}\big\{t\big(\lfloor (i+1)/2 \rfloor + 1\big) + t\big(k - \lfloor (i+1)/2 \rfloor\big) + i\big\}\Big\}.$$

Using the induction hypothesis, it can be shown that the right-hand side is at most $(2k-3)\lg k - 1$. This completes the induction. Thus, in all parallel linear searches we check at most $O(r \log r)$ terminals for a given direction. Clearly the time to check the first set of statements in the generalized version of Theorem 9 is at most a constant times the maximum possible degree of any node in an MST. It is well known that this degree is six [6] and since we need to check $O(\lambda)$ directions, the total time spent on removing safe regions in R is $O(\lambda r \log r)$. □

Theorem 12. *The algorithm presented above has $O(\lambda n)$ space requirement.*

Proof. MST M and CH(Z) require $O(n)$ storage. Since the (clockwise) walk of M has length $O(n)$ we can represent all paths of terminals encountered in the algorithm using a total of $O(n)$ space. Each terminal has $O(\lambda)$ d-visible terminals and edges. The space requirement for all calls to the sweep line algorithm is $O(n)$. Clearly, we can represent λ-SH(Z) using $O(n)$ space. This shows that the entire algorithm uses $O(\lambda n)$ space. □

Theorem 13. *For constant λ, the algorithm presented above is optimal.*

Proof. Clearly, any algorithm that computes λ-SH(Z) must use $\Omega(n)$ space. The terminals of Z on the boundary of the initial λ-Steiner hull λ-CH(Z) of the algorithm are exactly the terminals belonging to the boundary of CH(Z). Since these terminals remain on the boundary of the partially constructed λ-Steiner hull throughout the course of the algorithm, the boundary of λ-SH(Z) must also contain all terminals belonging to the boundary of CH(Z).

Clearly, the boundary of λ-SH(Z) is a simple polygon with $O(n)$ vertices. Since the convex hull of the vertices on a simple polygon with $O(n)$ vertices can be determined in $O(n)$ time [5], it follows that any algorithm computing λ-SH(Z) uses $\Omega(n \log n)$ time. □

8. Uniqueness of λ-SH(Z)

In this section, we show that λ-SH(Z) is uniquely defined in the sense that it does not depend on the chosen maximal sequence of safe removals from λ-CH(Z). The uniqueness proof is quite similar to that in [7] for the Steiner hull in the Euclidean metric.

We let $C(z_{i_1}, z_{i_2})$ denote the path of terminals encountered when walking along the boundary of λ-SH(Z) starting in z_{i_1} and ending in z_{i_2} where z_{i_1} and z_{i_2} belong to the same MST region. We need the following two lemmas.

Lemma 14. *Let R be a subregion induced by terminals $z_{i_1}, \ldots, z_{i_2}$. If z_k is a base terminal in R then $z_k \in C(z_{i_1}, z_{i_2})$.*

Proof. Suppose the lemma does not hold. Then there exists a maximal sequence of safe removals from R such that λ-SH(Z) contains a subregion R' in R induced by terminals $z_{j_1}, \ldots, z_{j_2}$ where $j_1 < k < j_2$. Let S be the safe region in R bounded by z_{i_1}, z_{i_2}, and z_k and let W be the λ-wedge bounding S. Then W is safe in R', a contradiction. □

If k is the smallest index such that z_k is a base terminal in subregion R then the removal of the corresponding safe region in R is called *canonical*. A maximal sequence of safe removals from R is said to be *canonical* if all its safe removals are canonical.

Lemma 15. *If λ-SH(Z) is obtained by some maximal sequence of safe removals from λ-CH(Z) then the same polygon can be obtained by a canonical sequence.*

The proof of Lemma 15 is in Appendix A. We now present the main result of this section.

Theorem 16. *λ-SH(Z) does not depend on the chosen maximal sequence of safe removals from λ-CH(Z).*

Proof. This follows from Lemma 15 and the fact that every canonical sequence of safe removals from λ-CH(Z) yields the same λ-SH(Z). □

9. Concluding remarks

In this paper, we defined a region λ-SH(Z) known to contain every λ-SMT for Z. Letting $n = |Z|$, we presented an $O(\lambda n \log n)$ time and $O(\lambda n)$ space algorithm that computes this set by removing open wedge-shaped regions from an initial hull. We proved that our algorithm is optimal in both time and space for constant λ and showed that λ-SH(Z) is independent of the order of removals of open wedge-shaped regions.

A possible improvement to the algorithm would be to flip suitable critical paths of λ-SH(Z). This would yield a smaller hull (which would not contain every λ-SMT but at least one) but it would not increase the number of terminals on the boundary of the hull. However, it would restrict the feasible locations of Steiner points further thus possibly making it easier to compute a λ-SMT for Z.

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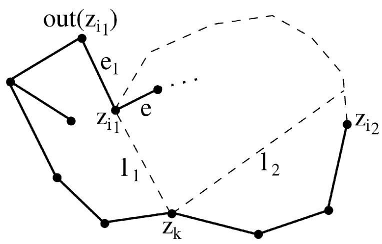

Fig. A.1. The situation in the proof of the generalized version of Theorem 9.

Appendix A. Theorem 9 generalized

In Theorem 9 we assumed that terminals were in general position. In this section we show how to remove this restriction. So suppose that the terminals of Z have arbitrary locations. We claim that, with the definitions in Theorem 9, the following two sets of statements are equivalent.

(1) z_1 is not d_2-visible from z_k and z_2 is not d_1-visible from z_k.
For $m = 1, 2$, if an edge e in R is d_m-visible from z_k then l_m crosses e from the outside of R looking from z_k.
For $m = 1, 2$, if a terminal $z_j \notin \{z_{i_1}, z_{i_2}\}$ in R is d_m-visible from z_k then l_m crosses z_j from the outside of R looking from z_k.
If z_{i_1} is d_1-visible from z_k then $W(z_{i_1}, z_k, out(z_{i_1}))^\circ$ contains no edges ending in z_{i_1}.
If z_{i_2} is d_2-visible from z_k then $W(z_{i_2}, in(z_{i_2}), z_k)^\circ$ contains no edges ending in z_{i_2}.
(2) $W(l_2, l_1)$ is a safe λ-wedge in R.

Assume that (1) is satisfied. To show (2) we only consider the cases not covered by Theorem 9. Letting p_m be as in the proof of Theorem 9, suppose for the sake of contradiction that p_m is a terminal z_j. The third statement in (1) implies that $z_j \in \{z_{i_1}, z_{i_2}\}$. If $z_j = z_{i_1}$ then l_1 must intersect z_{i_1} and z_{i_1} is d_1-visible from z_k.

Let e_1 be the edge directed from z_{i_1} to $out(z_{i_1})$. The right side of e_1 looking from z_{i_1} belongs to the outside of R. Since the interior of $z_k z_{i_1}$ contains no terminals then, as shown in Fig. A.1, e must intersect the interior of $W(z_{i_1}, z_k, out(z_{i_1}))$, contradicting (1). We show $z_j \neq z_{i_2}$ similarly.

We can exclude the situation in Fig. 8 since z_1 is not d_2-visible from z_k and z_2 is not d_1-visible from z_k. Thus, (1) $\Rightarrow$ (2).

Now suppose that (2) holds. Since l_1 is the left leg of W, l_1 cannot intersect z_{i_1} and l_2 cannot intersect z_{i_2}. For $m = 1, 2$ we have, by Lemma 8, that if a terminal $z_j \notin \{z_{i_1}, z_{i_2}\}$ in R is d_m-visible from z_k then l_m crosses z_j from the outside of R looking from z_k.

Finally suppose that say z_{i_1} is d_1-visible from z_k. No edges of R can cross the safe region in R bounded by $W(l_2, l_1)$. In particular, no edges in R with endpoint in z_{i_1} can cross this safe region. It follows that $W(z_{i_1}, z_k, out(z_{i_1}))^\circ$ contains no edges with endpoint in z_{i_1}. This shows (2) $\Rightarrow$ (1).

Appendix B. Proof of Lemma 15

Let S be any maximal sequence of safe removals from λ-CH(Z) and suppose that the first safe removal in S of some safe region R, bounded by z_{i_1}, z_{i_2}, and some base terminal, is not canonical. We will show that we can substitute R by the canonical removal of a safe region R' bounded by z_{i_1}, z_{i_2}, and a base terminal z_k, followed by an appropriate maximal sequence of safe removals such that λ-SH$'(Z)$ remains the same. By repeating this procedure a sufficient number of times, the lemma follows.

By Lemma 14, $z_k \in C(z_{i_1}, z_{i_2})$ when applying sequence S. Let S' denote the subsequence of S beginning with the removal of R and ending with z_k being added to the boundary.

Let S'' denote the subsequence of S' consisting of the safe removals of regions bounded by terminals of the form z_{j_1}, z_{j_2}, and some base terminal where $i_1 \leqslant j_1 < k < j_2 \leqslant i_2$. The sequence S'' followed by the sequence $S' \setminus S''$ yields the same boundary as S'. Hence, we may assume that S'' is a prefix of S'.

We start by removing R'. This splits our MST region into a subregion R_1 containing $z_{i_1}, \ldots, z_k$ and a subregion R_2 containing $z_k, \ldots, z_{i_2}$. Suppose $C(z_{i_1}, z_k)$ has an intermediate terminal and let z_h be the successor of z_{i_1} in $C(z_{i_1}, z_k)$. There is a safe region $\bar{R}$ bounded by z_{i_1}, z_j, and base terminal z_h for some $j \in \{k+1, \ldots, i_2\}$. If W is the λ-wedge bounding $\bar{R}$ then W is safe in R_1. We remove the corresponding safe region from R_1 and repeat the procedure on $C(z_h, z_k)$ if it has intermediate terminals.

We can apply the same procedure to $C(z_k, z_{i_2})$. Thus, we have modified our sequence S into another sequence starting with a canonical removal without affecting the resulting λ-$\mathrm{SH}'(Z)$.

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